Probability and Statistics - ENSAE
Example of Entrance Exam

PLEASE DO NOT TURN THIS PAGE OR START THE EXAM UNTIL YOU ARE INSTRUCTED TO DO SO.

1. Exam duration: 1.5 hrs
2. This is a closed book, closed notes exam. Calculators are permitted.
3. Make sure to mark your name at the top of this page and on the first page of all the answer sheets you use.
4. **Show detailed answers to receive full credit unless instructed otherwise.**

Problem 1	4	
Problem 2	3	
Problem 3	6	
Problem 4	7	
TOTAL	20	

Problem 1. (4 pts)

Let U and V be two i.i.d. uniform random variables in $[0, 1]$.

(a) Determine the distributions of $2U$ and $U + V$.

(b) Let $X = \min(U, V)$ and $Y = \max(U, V)$.

1. Find the distributions of X and Y.
2. Show that the pair (X, Y) has a density and compute it.

Problem 2. (3 pts)

Recall that for all $\lambda > 0$, the Poisson distribution with parameter λ is the distribution of an integer valued random variable X such that

$$\mathbb{P}[X = k] = C(\lambda)\frac{\lambda^k}{k!}, \quad \forall k \in \{0, 1, 2, \ldots\}$$

where $C(\lambda)$ is a positive constant.

(a) Show that $C(\lambda) = e^{-\lambda}$.

(b) For each positive integer n, let X_n be a random variable with the Poisson distribution with parameter $1/n$. Show that for any sequence $(a_n)_{n\geq 1}$ of positive numbers diverging to $+\infty$, $a_n X_n$ goes to zero in probability.

Problem 3. (6 pts)

Let $\theta > 0$ and $X_1, X_2, \ldots$ be i.i.d. random variables with the uniform distribution on $[0, \theta]$. For all positive integers n, let $M_n = \max(X_1, \ldots, X_n)$.

(a) Compute the cumulative distribution function of $n(\theta - M_n)$.

(b) Prove that M_n goes to θ in probability.

(c) Show that $n(\theta - M_n)$ converges in distribution, and identify the limiting distribution.

(d) Using the previous questions, compute one confidence interval for θ with non-asymptotic level 5%, and one confidence interval for θ with asymptotic level 5%.

Problem 4. (7 pts)

Let X be a non-negative random variable. Assume that X is *lacking memory*, that is, $\mathbb{P}[X > t] > 0$ for all $t > 0$ and

$$\mathbb{P}[X > s + t | X > s] = \mathbb{P}[X > t],$$

for all $s, t > 0$. Set $G(t) = \mathbb{P}[X > t]$, for all $t \geq 0$.

(a) Show that G is non-increasing and right-continuous (i.e., for all $t \geq 0$, and for all non-increasing sequences $(t_n)_{n \geq 1}$ converging to t, $G(t_n) \xrightarrow[n \to \infty]{} G(t)$).

(b) Show that for all $s, t \geq 0$, $G(s + t) = G(s)G(t)$.

(c) Conclude that for all $s \geq 0$ and for all non-negative integers k, $G(ks) = G(s)^k$.

(d) Show that for all positive integers q, $G(1/q) = G(1)^{1/q}$.

(e) Conclude that for all non-negative rational number r, $G(r) = G(1)^r$.

(f) Prove that for all non-negative real numbers t, $G(t) = G(1)^t$.

(g) What is the distribution of X ?